Challenge Journal of Structural Mechanics

Case Study

Girkmann problem with a Discrete Element Method

Christian Mariotti *

CEA, DAM, DIF, F.91297 Arpajon, France

ABSTRACT

Cells of Voronoï are used as particles in the Discrete Element code CeaMka3D. This type of meshing does not leave geometrical space like that can be the case with spherical particles. This method has already been used successfully to simulate the propagation of seismic waves in a linear elastic medium in 2D or in 3D. In this paper, a specific axisymmetric formulation is presented. In a first part, the calculation of the volumetric deformation of a particle and the forces between particles are described. In a second part, the specific forces for the axisymmetric formulation are described. At last, this formulation is tested for the Girkmann problem. This axisymmetric benchmark has been presented in January 2008 by the International Association of Computational Mechanics (IACM) in order to test the singularity at the junction between shell and beam. The accuracy of the axisymmetric formulation for this Discrete Element Method is evaluated by this benchmark. The results of this Discrete Element Method are compared with others numerical methods.

ARTICLE INFO

Article history:
Received 24 June 2016
Accepted 3 August 2016

Keywords:
Elasticity
Meshless
Axisymmetric
Discrete Element Method

1. Introduction

Particle methods are meshless simulation techniques in which a continuum medium is approximated through the dynamics of a set of interacting solids. These include the Discrete Element Methods (DEM) first developed by Hoover and al. (1974) in models for crystalline materials. They were applied to geotechnical problems by Cundall and Strack (1979).

A Discrete Element code CeaMka3D has been developed by Mariotti and Monasse (2012). This method has been used successfully to simulate, for example, the propagation of seismic waves in a linear elastic medium by Mariotti (2007). This code has also been coupled with a finite element code by Mariotti et al. (2015) and a new symplectic leapfrog scheme has been developed in order to integrate 3D rigid-body rotation with external torque by Mariotti (2015).

After a determination of the volumetric strain in 3D, the axisymmetric forces are described in order to recover Hooke's law. In the last part, the simulation with CeaMka3D of the Girkmann problem is presented. This problem has been described by Girkmann (1956) and studied by Timoshenko and Woinowsky-Krieger (1959). This axisymmetric benchmark has been presented in January 2008 by the International Associadtion of Computational Mechanics (IACM) in order to test the singularity at the junction between shell and beam. It has been studied as a benchmark problem by Pitkäranta et al. (2008) and Devloo et al. (2013).

2. The Volumetric Strain in 3D

The initial choice was to take a Voronoï mesh which allows, from a field of points, to bound polyhedrons. This type of meshing does not leave geometrical space like that can be the case with spherical particles. By geometrical construction, the plan of contact between two particles is perpendicular to the line connecting the centers of particles.

For example; let n be the normal direction between particles A and B as shown in Fig. 1.

$$n = \frac{AB}{\|AB\|} \ . \tag{1}$$

* *Corresponding author.* Tel.: +33-169-264000 ; Fax: +33-169-267130 ; E-mail address: christian.mariotti@cea.fr (C. Mariotti)
ISSN: 2149-8024 / DOI: http://dx.doi.org/10.20528/cjsmec.2016.08.017

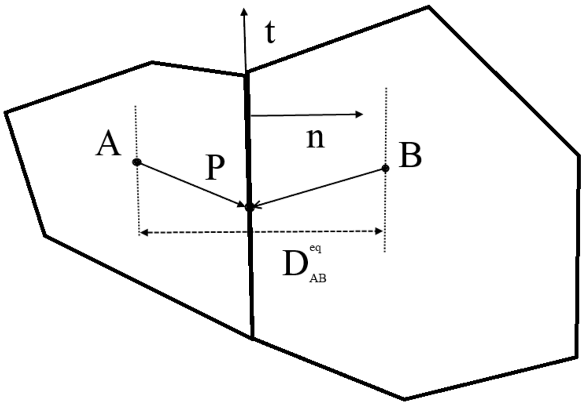

Fig. 1. Initial contact between two particles.

The initial distance between particles A and B at time t=0 is defined by:

$$D_{AB}^{eq} = \|AB\|_{t=0} \, . \quad (2)$$

The relative movement of both particles A and B according to the normal is defined by:

$$D_{AB}^{n} = D_{AB}^{eq} - \|AB\| \, . \quad (3)$$

It is necessary to choose the method of calculation of the elastic volumetric strain of a particle A surrounded by other particles B during their movements.

A particle A has only a part of its surface in touch with the other particles B. A volume of contact is defined by the following relation:

$$V_A^c = \frac{1}{3} \sum_{links\ B} \frac{1}{2} D_{AB}^{eq} S_{AB} \, , \quad (4)$$

where S_{AB} is the contact area between particles A and B.

The variation of this volume of contact is given by the following equation:

$$\Delta V_A^c = \sum_{links\ B} \frac{1}{2} D_{AB}^{n} S_{AB} \, . \quad (5)$$

To transform this variation of elastic volume of contact to the elastic volumetric strain, it is necessary to integrate free surfaces of the particle which can also have an elastic strain.

The complementary volume, called free volume, is then defined by:

$$V_A^F = V_A - V_A^c \, . \quad (6)$$

The variation of elastic volume of the particle A is given by the relation:

$$\varepsilon_A^V = \varepsilon_A^{VC} + \varepsilon_A^{VF} = \frac{\Delta V_A^C + \Delta V_A^F}{V_A} \, . \quad (7)$$

To define the variation of free volume, it is necessary to return to the Hooke's law. On the free surface and according to the normal for this free surface, the normal constraint is nil, so:

$$\varepsilon_A^{nn} = \frac{-\vartheta}{1-2\vartheta} \varepsilon_A^V \quad (8)$$

where ϑ is Poisson's ratio.

On the other hand, the normal elastic strain of the free surface is connected with the variation of the free volume

$$\varepsilon_A^{nn} = \frac{1}{3} \frac{\Delta V_A^F}{V_A^F} \, . \quad (9)$$

So:

$$\varepsilon_A^{VF} = \frac{\Delta V_A^F}{V_A} = -3 \frac{\vartheta}{1-2\vartheta} \left(\frac{V_A^F}{V_A} \right) \varepsilon_A^V = -3 \frac{\vartheta}{1-2\vartheta} \left(1 - \frac{V_A^C}{V_A} \right) \varepsilon_A^V \, , \quad (10)$$

and

$$\varepsilon_A^{VC} = \frac{\Delta V_A^C}{V_A} \, . \quad (11)$$

So:

$$\varepsilon_A^V = \varepsilon_A^{VC} + \varepsilon_A^{VF} = \frac{\Delta V_A^C}{V_A} - 3 \frac{\vartheta}{1-2\vartheta} \left(1 - \frac{V_A^C}{V_A} \right) \varepsilon_A^V$$

$$= \frac{\Delta V_A^C}{V_A} \frac{1}{1+3\frac{\vartheta}{1-2\vartheta}\left(1-\frac{V_A^C}{V_A}\right)} \, . \quad (12)$$

At the end,

$$\varepsilon_A^V = \frac{1}{V_A} \frac{1}{1+3\frac{\vartheta}{1-2\vartheta}\left(1-\frac{V_A^C}{V_A}\right)} \sum_{links\ B} \frac{1}{2} S_{AB} D_{AB}^n \, . \quad (13)$$

An expression for the calculation of the elastic volumetric strain of a particle A has been determined. In the expression of the normal force between particles A and B, the following volumetric strain will be used.

$$\varepsilon_{AB}^V = \frac{1}{2} (\varepsilon_A^V + \varepsilon_B^V) \, . \quad (14)$$

The normal force between both particles A and B is then given by:

$$F_{AB}^n = \left(K_S \frac{D_{AB}^n}{D_{AB}^{eq}} + K_V \varepsilon_{AB}^V \right) S_{AB} \, , \quad (15)$$

with

$$K_S = \frac{E}{1+\vartheta} \, , \quad (16)$$

$$K_V = \frac{E\vartheta}{(1+\vartheta)(1-2\vartheta)} \, , \quad (17)$$

where E and ϑ are respectively Young's modulus and Poisson's ratio.

The axisymmetric formulation is examined in the next part.

3. 2D Axisymmetric Formulation

In a cylindrical coordinate system (r, θ, z), some problems are independent with respect to coordinate θ. This symmetry enables reduction to a 2D axisymmetric problem. The expression of Hooke's law gives:

$$\sigma_{rr} = \frac{E}{1+\vartheta}\frac{\partial u_r}{\partial r} + \frac{E\vartheta}{(1+\vartheta)(1-2\vartheta)}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right),$$

$$\sigma_{\theta\theta} = \frac{E}{1+\vartheta}\frac{u_r}{r} + \frac{E\vartheta}{(1+\vartheta)(1-2\vartheta)}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right),$$

$$\sigma_{zz} = \frac{E}{1+\vartheta}\frac{\partial u_z}{\partial z} + \frac{E\vartheta}{(1+\vartheta)(1-2\vartheta)}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right),$$

$$\sigma_{rz} = \frac{E}{2(1+\vartheta)}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),$$

$$\sigma_{r\theta} = 0\,,$$

$$\sigma_{z\theta} = 0\,. \tag{18}$$

The kinematic of the particles is reduced to three degrees of freedom(u_r, u_z, θ). The geometry of the particle A is defined by a thickness equal to the value of its coordinate r at the initial time r_A^0. The particle is defined by an angle portion of 1 radian. The volumetric strain of a particle can be separated into two terms. The first term corresponds to the volumetric strain in plane (r, z), which is used to define a volume change ε_{AB}^V of particle A with its neighbours B in plane (r, z) like in the previous part. The second term in u_r/r corresponds to the displacement of a particle A along the radial direction r.

The stress term $\sigma_{\theta\theta}$ introduces an additional force along axis r. This force is calculated assuming that particle A has two neighbouring particles in the orthoradial direction θ. Stress $\sigma_{\theta\theta}$ can be rewritten in the form below:

$$\sigma_{\theta\theta} = \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)}\frac{u_r}{r} + \frac{E\vartheta}{(1+\vartheta)(1-2\vartheta)}\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right). \tag{19}$$

The corresponding force $F_{\theta\theta}^A$ is orientated along the radial direction r of the plane, surface S_A corresponds to the surface of particle A in plane (r, z), the equilibrium distance with the two orthoradial virtual particles corresponds to the initial value r_A^0 of r for particle A, and the volumetric strain of particle A with its neighbours, ε_{AB}^V, is calculated in plane (r, z) only.

$$F_{\theta\theta}^A = \frac{E(1-\vartheta)S_A}{(1+\vartheta)(1-2\vartheta)}\frac{r_A^0 - r_A}{r_A^0} + \frac{E\vartheta S_A}{(1+\vartheta)(1-2\vartheta)}\,\varepsilon_{AB}^V\,. \tag{20}$$

This formulation has already been verified for various value of the Poisson's ratio by Mariotti and Monasse (2012), but this formulation for the Girkmann problem was verified in this study.

4. Girkmann Problem

The Girkmann problem is a benchmark for an axisymmetric shell supported on a stiffening ring. The junction between the shell and the beam is a singularity which may output some deficiencies for a numerical method. This test was first described by Girkmann (1956) and later by Timoshenko and Woinowsky-Krieger (1959). A Benchmark was proposed in January 2008 by the International Association of Computation Mechanics (IACM).

This benchmark has been described in a paper from Szabo et al. (2010) and a paper from Devloo et al. (2013). A spherical shell of thickness h = 0.06 m and crown radius R_c = 15 m is connected to a stiffening ring at the meridional angle α = 40°. The middle radius of the spherical shell is $R_m = R_c/\sin\alpha$. The dimension of the section of the ring are horizontally a = 0.6 m and vertically b = 0.5 m (Fig. 2).

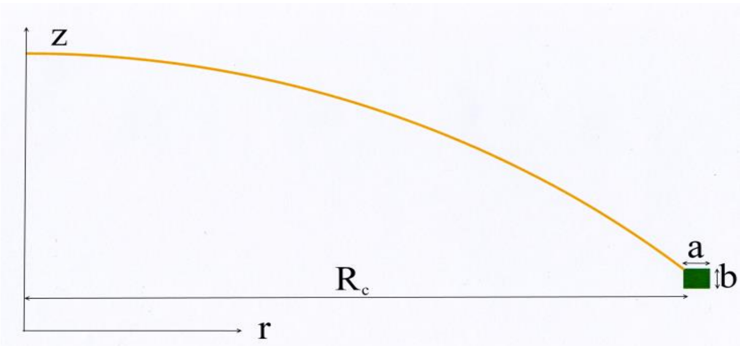

Fig. 2. Geometry of the Girkmann problem (Shell in yellow, Ring in green).

The shell is elastic with a Young modulus E = 20.59 10^9 Pa and a Poisson's ratio ϑ = 0. The density of the material is 3,269 kg/m^3. A gravity force of 10 m/s^2 is applied on the shell only and not on the ring. A uniform vertical pressure is acting on the base of the stiffening ring in order to equilibrate the gravity force. The values expected in this benchmark are the shearing force Q in N/m and the bending moment M in Nm/m acting at the junction between the spherical shell and the stiffening ring.

Classical continuous Finite Element Method may encounter difficulties at the singularity between the beam and the shell. For example, several results of axisymmetric Finite Element Method are presented in the paper of Szabo et al. (2010), the values of Q are varying between 940.9 N/m and 989.1 N/m and the values of M are varying between -36.62 and -89.11 Nm/m.

According to Devloo et al. (2013), the Discontinuous Galerkin method has been shown more accurate in solving this type of singularity. So, the values of Q and M given by Devloo et al. (2013) for the high order DG formulation are taken as reference results for this benchmark (Table 1).

In this simulation with the Discrete Element Method, the shell and the ring are meshed with particles of about 0.005 m as shown in Fig. 3. The CeaMka3D simulation gives comparable results with the results of Devloo et al. (2013) (Table 1), the relative error is less than 0.35 %. So, the Discrete Element Method gives accurate results for this benchmark.

Table 1. Results for the Girkmann problem.

Numerical Method	Q (N/m)	M (Nm/m)
Pitkäranta Classical M-B-R model	942.5	-37.45
Devloo DG-FEM	943.65	-36.79
CeaMka3D DEM	946.4	-36.66

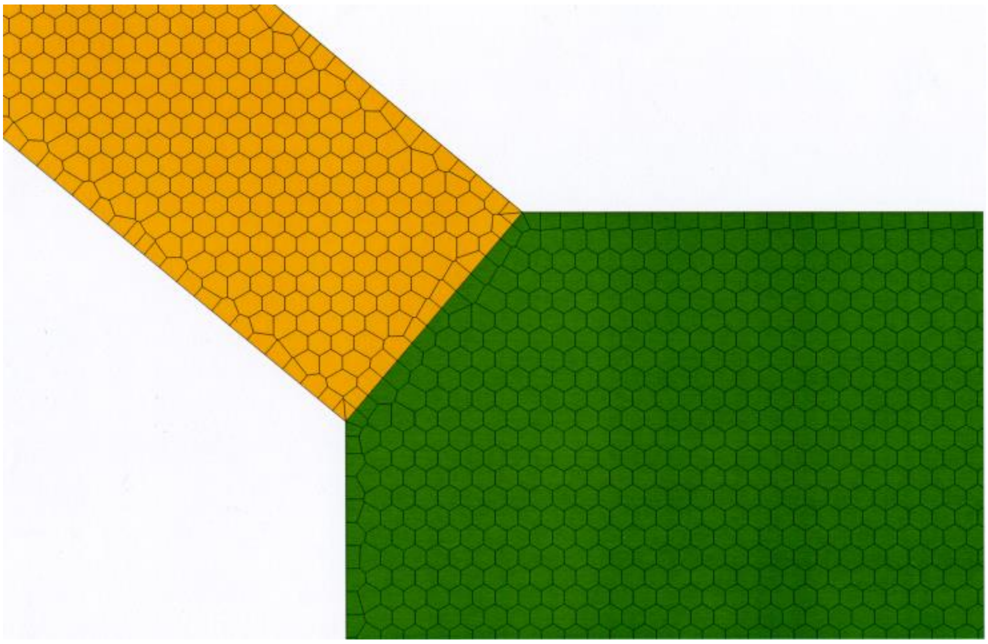

Fig. 3. Zoom of the mesh at the junction between the shell and the stiffening ring for CeaMka3D (Shell in yellow, Ring in green).

5. Conclusions

In the first part, the calculation of the volumetric deformation of a particle and the definition of forces and torques between particles has been described.

In the second part, an axisymmetric formulation for the Discrete Element code CeaMka3D has been presented. This formulation is applied to verify the robustness of the Discrete Element Method to the singularity of the Girkmann problem. This benchmark has been chosen because it may be quite difficult for classical continuous Finite Element Method.

The results given by the Discrete Element Method are closed to the reference results given by a High Order Discontinuous Galerkin Method. So this Discrete Element Method seems accurate for this kind of benchmark with a strong gradient. The axisymmetric formulation for this Discrete Element Method is also verified by this benchmark. It is an encouraging result for the development of this kind of method.

The Voronoï mesh has been chosen initially because this type of meshing does not leave geometrical space like that can be the case with spherical particles and the normal direction is linked directly to the positions of particles. In fact, the volume of contact of each particle is very important in order to define the volumetric strain.

But at present, one limitation of this Discrete Element formulation is the Voronoï mesh. In fact, a special treatment is necessary to create a Voronoï mesh along interfaces. It has been necessary to create a specific mesh generator in 2D and 3D in order to respect some geometrical interfaces with Voronoï mesh. In the future, it will be easier to use classical mesh generator with triangles particles in 2D or tetrahedrons particles in 3D. So, one direction for the development of this Discrete Element Method will be to define the forces and torques for two triangles or two tetrahedrons in contact.

References

Cundall PA, Strack ODL (1979). A discrete numerical model for granular assemblies. *Geotechnique*, 29(1), 47-65.

Devloo PRB, Farias AM, Gomes SM, Gonçalves JL (2013). Application of a combined continuous-discontinuous Galerkin finite element method for the solution of the Girkmann problem. *Computers and Mathematics with Applications*, 65, 1786-1794.

Girkmann K (1956). Flächentragwerke. 4th ed. Springer-Verlag, Wien.

Hoover WG, Arhurst WT, Olness RJ (1974). Two-dimensional studies of crystal stability and fluid viscosity. *Journal of Chemical Physics*, 60, 4043-4047.

Mariotti C (2007). Lamb's problem with the lattice model Mka3D. *Geophysical Journal International*, 171, 857-864.

Mariotti C (2015). A new Leapfrog scheme for rotational motion in 3D. *International Journal for Numerical Methods in Engineering*, 107, 273-289.

Mariotti C, Monasse L (2012). From General Mechanics to Discontinuity, Unified Approach to Elasticity. Presses des Ponts, France.

Mariotti C, Le Piver F, Aubry L (2015). A least-squares coupling method between a finite element code and a discrete element code. *International Journal for Numerical Methods in Engineering*, 101(10), 731-743.

Monasse L, Mariotti C (2012). An energy-preserving Discrete Element Method for elastodynamics. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(6), 1527-1553.

Pitkäranta J, Babuska I, Szabo B (2008). The Girkmann problem. *IACM Expression*, January 2008, 22-28.

Szabo B, Babuska I, Pitkäranta J, Nervi S (2010). The problem of verification with reference to the Girkmann problem. *Engineering with Computers*, 26, 171-183.

Timoshenko SP, Woinowsky-Krieger S (1959). Theory of Plates and Shells. McGraw-Hill, New York.