CHALLENGE JOURNAL OF STRUCTURAL MECHANICS 7 (1) (2021) 1–10

Challenge Journal of Structural Mechanics

Research Article

Determination of the exact mode frequencies of multi-storey structures by state-space method and a comparison with mode superposition method

Ahmad Yamin Rasa [a,*], Mehmet Hamit Özyazıcıoğlu [a]

[a] Department of Civil Engineering, Atatürk University, 25240 Erzurum, Turkey

ABSTRACT

In this paper, a comparative research obtaining the time-consuming exact solution and approximate solution of linearly damped linear frame buildings has been carried out. State-space method and mode superposition method are respectively utilized to analyze the exact and approximate vibration solution of frame buildings. In mode superposition method, the proportional damping matrix of frame buildings has been constructed in two different schemes, such as modal combination of mass with stiffness matrices (Rayleigh) and disregarding the off-diagonal elements of non-classical damping (DOEA) matrix. Meanwhile, the damping matrix in state-space method is non-proportional and constructs in exact circumstance. By imposing different initial conditions on the system, it has been proved that the transient vibration is mostly greater than the steady-state vibration. These observations have individually been investigated to render the approximate response as close as possible to the exact response. The findings of current research indicate that the most suitable parameter which brings closer the approximate response to the exact response is DOEA approach. In addition, a MATLAB code, which solves the equation of exact and approximate method, is generated in this research.

ARTICLE INFO

Article history:
Received 13 July 2020
Revised 28 September 2020
Accepted 23 October 2020

Keywords:
Structural dynamics
State-space method
Mode superposition method
Transient vibration
Mode frequency

1. Introduction

In structural dynamics, mode superposition method is a general approach, which can analyze transient and steady-state response of structures under different dynamic loads. Modal frequencies and modal vectors of structure are generally calculated in un-damped situation. Based on the idea of Borino and Muscolin (1986), exact frequencies (damped frequencies) of structures will not differ much from un-damped frequencies. This idea is clearly explained with examples in this article. In this method, damping matrix could be generated either by linear combination of mass and stiffness matrix (proportional damping) or by neglecting the non-diagonal elements of non-proportional damping matrix, Felszeghy (1986), Wilson and Penzien (1972). This approximate method is generally implemented in three stages; 1) evaluating the modal frequencies and modal vectors of un-damped system, 2) creating the proportional damping matrix of the system and 3) combining the modal responses. The maximum responses (displacement, velocity, acceleration or internal forces) can be calculated either by Square Root of Sum of the Squares (SRSS) method or Complete Quadratic Combination (CQC) method. These methods are exist in Sinha and Igusa (1992), Wilson, Kiureghian and Bayo (1981) and Zhou, Yu and Liang (2004) articles.

In state-space method, which gives the exact response of dynamic systems, damping matrix constructs in exact situation and is always non-proportional. Mode frequencies and mod vectors of the system are always in complex conjugate form; and modal frequencies are damped frequencies, not same as mod superposition method. It would be better to predict the modal damping factors of

* *Corresponding author*. E-mail address: ahmadyamin.rasa19@ogr.atauni.edu.tr (A. Y. Rasa)
ISSN: 2149-8024 / DOI: https://doi.org/10.20528/cjsmec.2021.01.001

each mode form damped frequencies, Veletsos and Ventura (1986).

In this research, the difference between exact and approximate response of a vibrational structure has been investigated in detail. The most suitable type of proportional damping matrix, which gives the nearest response to the exact response, is determined. Proportional damping matrix is constructed in three types; (a) combining first and second modes of vibration (MSM-a), (b) combining first and last modes of vibration (MSM-b) and (c) combining first and third modes of vibration in five story frame building (MSM-c). Modal damping ratios of frame buildings are derived from exact damped frequencies of them. In practical analysis, damping ratio of the whole system is approximately given such 5% ratio (MSM-5%); the result of this condition is investigated in detail. One another type of damping matrix, which is simplified by disregarding the off-diagonal elements approach (DOEA), is also observed in this research paper. Harmonic forces are applied on different floors of three and five storey frame buildings. Dynamic responses of frame buildings are comparatively displayed using figures and tables. Maximum responses are calculated by square root of sum of the squares (SRSS) method, Sinha, Igusa (1992). Caughey damping is not utilized in this research which are based on Caughey (1960) and Caughey and O'Kelly (1965) articles.

2. Methods

2.1. Analytical formulation of mode superposition method

Consider a classical damped linear structure subjected to harmonic force $f(t)$. The equation of motion for this system, which structural displacements are relative to dynamic load, may be written as;

$$\mathbf{m}\underline{\ddot{u}} + \boldsymbol{c}\underline{\dot{u}} + \boldsymbol{k}\underline{u} = \underline{f}(t) \tag{1}$$

where, **m**, **c** and **k** respectively denotes mass matrix, damping matrix and stiffness matrix of structure-damper system. And $\underline{\ddot{u}}, \underline{\dot{u}}, \underline{u}, \underline{f}(t)$ respectively expresses acceleration vector, velocity vector, relative displacement vector and force vector. Each modal parameter of frame building, which is un-damped natural frequency ω_n and mode vibration $\underline{\emptyset}_n$, can be calculated by eigen-equation of,

$$-\omega_n^2 \boldsymbol{m}\underline{\emptyset}_n + \boldsymbol{k}\underline{\emptyset}_n = 0 \tag{2}$$

If it is assumed that the matrix of orthonormal modes $\emptyset$ associated with the un-damped system can be used as a transformation matrix to uncouple Eqs. (1), then a set of generalized co-ordinates $\underline{q}$ may be defined like $u(t) = \underline{\emptyset}q(t)$. Therefore, generalized forces $F_n(t)$, masses M_n, stiffness constants K_n, damping constants C_n could be defined using transformation matrix $\emptyset$. Thus, the resulting equations are uncoupled if the damping matrix **c** is a linear combination of inertia matrix **m** and/or stiffness matrix **k.** Therefore, Eqs. (1) becomes like,

$$\boldsymbol{M}\underline{\ddot{q}} + \boldsymbol{C}\underline{\dot{q}} + \boldsymbol{K}\underline{q} = \underline{F}(t) \tag{3}$$

This equation consists of n uncoupled equations which may be solved by standard procedures. Thus, nth uncoupled equation of motion can written like,

$$M_n\ddot{q}_n + C_n\dot{q}_n + K_n q_n = F_n(t) \tag{4}$$

In which q_n is the generalized coordinate of the system. Modal damping ratio ζ_n which is belongs to the n'th mode can be calculated dividing Eq. (4) by M_n,

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega^2{}_n q_n = \frac{F_n(t)}{M_n} \tag{5}$$

Thus, modal damping ratio of any mode can be calculated by equation;

$$\zeta_n = \frac{C_n}{2M_n\omega_n} \tag{6}$$

Total solution of Eq. (5) may be written as vector sum of two components (free vibration + forced vibration),

$$q_n(t) = e^{-\zeta_n\omega_n t}\left[q_n(0)cos(\omega d_n t) + \frac{\dot{q}(0)+\zeta_n\omega_n q(0)}{\omega d_n} sin(\omega d_n t)\right] + \frac{1}{M_n\omega d_n}\int_0^t F_n(\tau)\, e^{-\zeta_n\omega_n(t-\tau)} sin[\omega d_n\ (t-\tau)]d\,\tau \tag{7}$$

At the right hand side of Eq. (7), the first term is called free vibration (homogeneous solution) part and second term is called forced vibration (non-homogeneous) part. More details about free vibration of structural elements could be found in Avcar (2104) and Civalek and Avcar (2020) articles. In this equation $q_n(0)$, $\dot{q}_n(0)$ and $\omega d_n = \omega_n\sqrt{1-\zeta_n^2}$ are initial displacement, initial velocity and damped frequency of the nth mode, respectively. Thus, displacement response of the nth mode may be written as,

$$\underline{u}_n(t) = \underline{\emptyset}_n q_n(t) \tag{8}$$

Displacement responses may be calculated combining all vibrating modes,

$$\underline{u}(t) = \sum_{n=1}^{N} \underline{\emptyset}_n q_n(t) \tag{9}$$

2.2. Analytical formulation of state-space method

In this method Eq. (1) which is a second-order differential equation has been reduced to a first-order differential equation consisting 2xN equations, based on Veletsos and Ventura (1986) and Villaverde (2008) articles. Substituting $\underline{\ddot{u}}$ to $\underline{\dot{z}}$ and $\underline{\dot{u}}$ to $\underline{z}$ then put these relations to Eq. (1),

$$\underline{\dot{u}} = \underline{z}$$

$$\boldsymbol{m}\underline{\dot{z}} + \boldsymbol{c}\underline{z} + \boldsymbol{k}\underline{u} = \underline{f}(t) \tag{10}$$

Then, new variables (displacement and velocity vectors) can be summed into vector $\underline{Z}$,

$$\underline{Z} = \begin{bmatrix} \underline{u} \\ \underline{z} \end{bmatrix}_{2n\times1} \tag{11}$$

After some mathematical operations on Eq. (10), Eq. (12) is obtained as,

$$\begin{bmatrix} \underline{\dot{u}} \\ \underline{\dot{z}} \end{bmatrix}_{2n\times1} = \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{I}_{n\times n} \\ -\boldsymbol{m}^{-1}\boldsymbol{k} & -\boldsymbol{m}^{-1}\boldsymbol{c} \end{bmatrix}_{2n\times2n} \begin{bmatrix} \underline{u} \\ \underline{z} \end{bmatrix}_{2n\times1} + \begin{bmatrix} 0 \\ \boldsymbol{m}^{-1}\underline{f}(t) \end{bmatrix}_{2n\times1} \tag{12}$$

Reduced form of Eq. (1), which is a first order differential equation, can be represented as a general form of,

$$\underline{\dot{Z}}(t) = \boldsymbol{A}\underline{Z} + \underline{P}(t) \tag{13}$$

In this equation, constant matrix $\mathbf{A}$, dynamic force vector $\underline{P}(t)$ and initial condition vector $\underline{Z}(0)$ can be expressed as,

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{I}_{n\times n} \\ -\boldsymbol{m}^{-1}\boldsymbol{k} & -\boldsymbol{m}^{-1}\boldsymbol{c} \end{bmatrix}_{2n\times2n} \tag{14}$$

$$\underline{P}(t) = \begin{bmatrix} 0 \\ \boldsymbol{m}^{-1}\underline{f}(t) \end{bmatrix}_{2n\times1} \tag{15}$$

$$\underline{Z}(0) = \begin{bmatrix} \underline{u}(0)_{n\times1} \\ \underline{\dot{u}}(0) \end{bmatrix} \tag{16}$$

The analytical solution of Eq. (13), which has homogeneous part and non-homogeneous part, is the same solution of Eq. (5).

2.2.1. Free vibration response

The dynamic force vector in Eq. (13) will be zero ($\underline{P}(t) = 0$) if the system vibrates freely. Thus, homogeneous form of Eq. (13) can be expressed as,

$$\underline{\dot{Z}}(t) = \boldsymbol{A}\underline{Z} \tag{17}$$

Solution of this equation may be written as,

$$\underline{Z}_h = \boldsymbol{e}^{\boldsymbol{A}t}\underline{c} \tag{18}$$

In this solution vector $\underline{c}$ can be determined by utilizing initial condition, so;

$$\underline{Z}(0) = \underline{c} \tag{19}$$

$$\underline{Z}_h = \boldsymbol{e}^{\boldsymbol{A}t}\underline{Z}(0) \tag{20}$$

Eq. (20) is the free vibration response of state-space Eq. (13).

2.2.2. Forced vibration response (general solution)

General solution of Eq. (13) can be found using variation of parameters. If homogeneous solution of Eq. (13) assumes as K, then the non-homogeneous solution of them can be written as;

$$\underline{Z}_p = \underline{V} \times \underline{K} \tag{21}$$

Parameter $\underline{V}$ can be calculated putting Eq. (21) in to Eq. (13). Therefore, the general solution of state-space Eq. (13) can be obtained like,

$$\underline{Z}(t) = \boldsymbol{e}^{\boldsymbol{A}t}\underline{Z}(0) + \boldsymbol{e}^{\boldsymbol{A}t}\int_0^t \boldsymbol{e}^{-\boldsymbol{A}\tau}\underline{Z}(0)\underline{P}(\tau)d\tau \tag{22}$$

or;

$$\underline{Z}(t) = \boldsymbol{e}^{\boldsymbol{A}t}\underline{Z}(0)\int_0^t \boldsymbol{e}^{\boldsymbol{A}(t-\tau)}\underline{Z}(0)\underline{P}(\tau)d\tau \tag{23}$$

In Eqs. (22) and (23), the first term at the right hand side denotes free vibration while the second term denotes forced vibration. Algebraic solution of Eq. (22) or (23) depends on the solution of $\boldsymbol{e}^{\boldsymbol{A}t}$. Therefore, utilizing the concepts of linear algebra, eigenvalues and eigenvectors of square matrix $\boldsymbol{A}$ can be calculated using eigen-equation bellow,

$$(\boldsymbol{A} - v_n\boldsymbol{I})\underline{x}_n = \underline{0}, \quad n = 1,2,3 \ldots \tag{24}$$

In this equation, $\boldsymbol{I}, \underline{v}_n$ and $\underline{x}_n$ are respectively unite square matrix, nth complex eigenvalue and eigenvector of squared matrix $\boldsymbol{A}$. This equation gives a set of $2 \times N$ eigenvalues v_n and eigenvectors $\underline{x}_n$, N is the number of storeys. These eigen-solutions always exist like complex conjugate form bellow, Veletsos and Ventura (1986) and Villaverde (2008);

$$\left.\begin{matrix} v_n \\ \overline{v}_n \end{matrix}\right\} = -e_n \pm is_n, \quad n = 1,2,3 \ldots \tag{25}$$

$$\left.\begin{matrix} \underline{x}_n \\ \overline{\underline{x}}_n \end{matrix}\right\} = -\underline{\varphi}_n \pm i\underline{\alpha}_n, \quad n = 1,2,3 \ldots \tag{26}$$

In the relations above, complex conjugate eigenvalues and eigenvectors, which respectively represent frequencies and modes of vibration, has been marked a dash on it. Real part of complex eigenvalue e_n representing damping mechanism of nth mode must be negative and, complex part of eigenvalue, which represents damped frequency of nth mode, can give damping factor of that mode. Thus, damping ratio for each mode of vibration can be calculated by equation,

$$\zeta_n = \frac{e_n}{\sqrt{e_n^2 + s_n^2}}, \qquad n = 1,2,3 \ldots \tag{27}$$

Based on Veletsos and Ventura (1986) article, there will be a unique damping ratio for any mode of vibration if Eq. (23) is utilized.

3. Results and Discussion

In this section, three and five storey frame buildings, which are taken from Rasa (2017), are selected as examples. The methods described above have been utilized in these examples and, the total responses are obtained by summation of free and forced vibration.

3.1. Example 1

For simplicity, only second floor responses are selected to compare the results here.

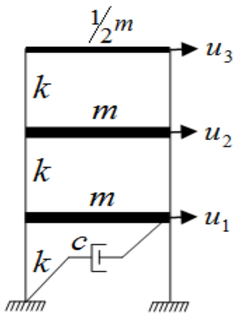

Fig. 1. Three storey structure with a damper at the bottom storey.

Initial displacement vector, initial velocity vector, dynamic force, inertia constant, stiffness constant and damping constant of the frame building (Fig. 1) has respectively been given as $\underline{u}(0) = \{0, 0, 0.1\}^T$ m, $\underline{\dot{u}}(0) = \{0, 0.5, 0\}^T$ m/sec , $f_1(t) = 0.1\sin(\pi t/0.3)$ kN, m=1 kg, k=1 kN/m and c=0.2 kNs/m. Therefore, mass matrix, damping matrix, stiffness matrix-and force vector, initial displacement vector and initial velocity vector of frame building can respectively be written as bellow;

$$\boldsymbol{m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{k} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \underline{f}(t) = \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{u}(0) = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} \qquad \underline{\dot{u}}(0) = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}$$

In Tables 1 and 2, complex mode frequencies and exact modal damping factors of frame building are respectively calculated by Eqs. (24) and (27). Creating proportional damping matrix of frame building needs a damping factor ζ, which is given approximately 5% and 0.033%. Taking the average of exact modal damping factors from Table 1 gives damping factor (0.033%).

Table 1. Exact mode frequencies and modal damping ratios of the frame building.

	1. Mode	2. Mode	3. Mode
ω_n	$-0.0163 \pm i1.9290$	$-0.0671 \pm i1.4126$	$-0.0167 \pm i0.5181$
ζ_n	0.0084	0.0474	0.0322

Table 2. Complex mode vectors of the frame building.

0.0340 – 0.1568i	0.0340 + 0.1568i	0.0195 + 0.4105i	0.0195 - 0.4105i	-0.3135 + 0.0188i	-0.3135 - 0.0188i
-0.0079 + 0.2813i	-0.0079 - 0.2813i	-0.0383 + 0.0000i	-0.0383 - 0.0000i	-0.5436 + 0.0054i	-0.5436 - 0.0054i
-0.0028 - 0.3269i	-0.0028 + 0.3269i	-0.0198 - 0.4032i	-0.0198 + 0.4032i	-0.6277 + 0.0000i	-0.6277 + 0.0000i
0.3019 + 0.0680i	0.3019 - 0.0680i	-0.5812 + 0.0000i	-0.5812 + 0.0000i	-0.0045 - 0.1628i	-0.0045 + 0.1628i
-0.5426 - 0.0198i	-0.5426 + 0.0198i	0.0025 - 0.0541i	0.0025 + 0.0541i	0.0063 - 0.2817i	0.0063 + 0.2817i
0.6306 + 0.0000i	0.6306 + 0.0000i	0.5709 - 0.0010i	0.5709 + 0.0010i	0.0105 - 0.3252i	0.0105 + 0.3252i

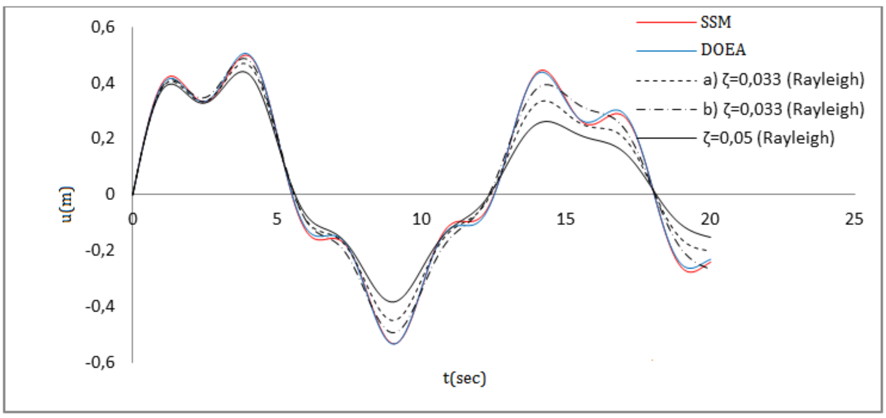

Fig. 2. Displacement response of second floor: (a) Rayleigh damping is created using first and second modes; (b) Rayleigh damping is created using first and third modes of vibration.

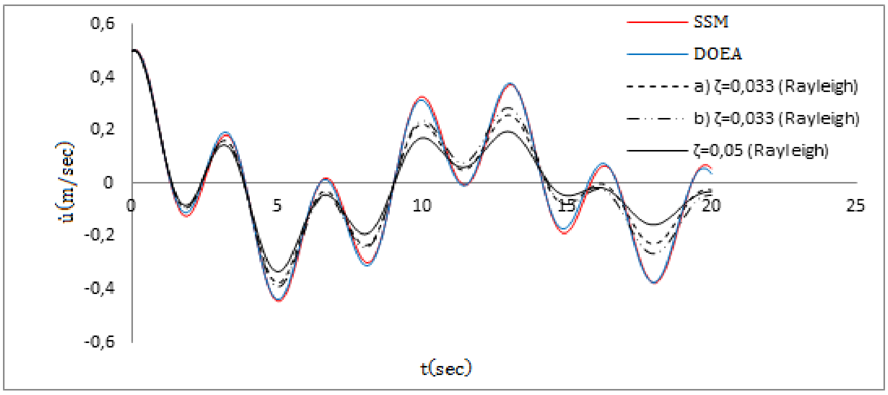

Fig. 3. Velocity response of second floor: (a) Rayleigh damping is created using first and second modes; (b) Rayleigh damping is created using first and third modes of vibration.

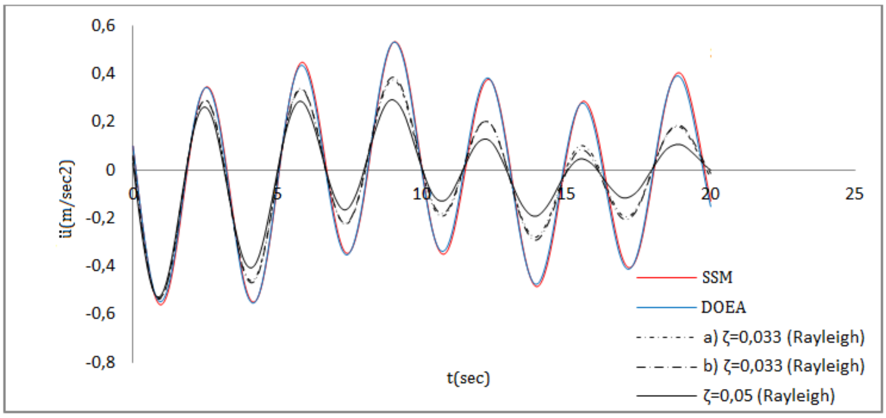

Fig. 4. Acceleration response of second floor:(a) Rayleigh damping is created using first and second modes; (b) Rayleigh damping is created using first and third modes of vibration.

According to Table 3, it is proved that DOEA method has a good agreement with SSM method. Among four approximate methods, the closest response to exact response is given by DOEA method. After DOEA method the closest response, which is in second order, is given by MSM-b method. Results of other two methods (MSM-a and MSM-5%) are not in good agreement with exact method. The maximum displacement response calculated by SSM and DOEA methods occur in 9.045 seconds, while by the other methods (MSM-a, MSM-b and MSM-5%) occur in 3.819 seconds. The displacement response in 3.819 seconds of exact method is seen to be 0.4983 meters.

Table 3. Maximum responses of the second floor.

Method	Time (sec)	Max. Dis. (m)	Time (sec)	Max. Vel. (m/sec)	Time (sec)	Max. Acce. (m/sec^2)
SSM	9.045 (3.819)	0.5332 (0.4983)	0	0.5	1.005 (4.121)	0.5587 (0.5473)
DOEA	9.045	0.5341	0	0.5	4.121	0.5522
MSM-a	3.819	0.4710	0	0.5	0.9045	0.5367
MSM-b	3.819	0.4887	0	0.5	0.9045	0.5320
MSM-5%	3.819	0.4413	0	0.5	0.9045	0.5280

In second mode of vibration, the exact un-damped natural frequency and the exact modal damping ratio of the frame building, as seen in Table 4 are respectively 1.4141 and 0.0474. In contrast, the results 1.4142 and 0.0471 are respectively very close to that of the exact method. Thus, the closest response to the exact response was calculated by DOEA method. Modal damping ratios calculated by other methods (MSM-a, MSM-b and MSM-5%) are not in good agreement with SSM method. As seen in Figs. 1, 2 and 3 there is a very good agreement between time history responses of SSM method and DOEA method.

As seen in Table 5, the velocity and acceleration responses of DOEA method are very close to that of the exact method, only the displacement response of MSM-b method is very close to that of exact method. It is also considered that the results of MSM-5% method has the most deviation among three methods (DOEA, MSM-a, MSM-b) with the rate of 11.14%. It can be concluded that DOEA method has very good agreement with SSM method at all.

Table 4. Modal damping ratios and mode frequencies of the second mode.

Method	ζ_2	ω_2	ωd_2
SSM	0.0474	1.4141	1.4126
DOEA	0.0471	1.4142	1.4126
MSM-a	0.0330	1.4142	1.4134
MSM-b	0.0286	1.4142	1.4136
MSM-5%	0.0500	1.4142	1.4124

Table 5. Maximum responses of the frame building.

Method	Max. Dis. (m)	Max. Vel. (m/sec)	Max. Accel. (m/sec²)
SSM	0.7486	0.5095	0.9638
DOEA	0.7495	0.5095	0.9624
MSM-a	0.7249	0.5081	0.8944
MSM-b	0.7487	0.5086	0.8936
MSM-5%	0.6975	0.5071	0.8564

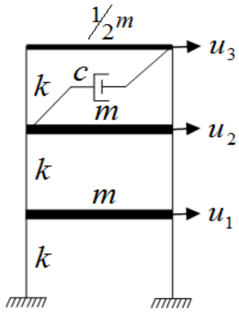

Fig.5. Three storey structure with a damper at the bottom storey.

$$\boldsymbol{c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.2 & -0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}$$

Creating proportional damping matrix of frame building needs a damping factor ζ, which is given approximately 5% and 0.056%. Taking the average of exact modal damping factors from Table 6 gives damping factor (0.056%).

Table 6. Exact mode frequencies and modal damping ratios of the frame building.

	1. Mode	2. Mode	3. Mode
ω_n	$-0.2374 \pm i1.8824$	$-0.0614 \pm i1.4383$	$-0.0012 \pm i0.5178$
ζ_n	0.1251	0.0427	0.0023

3.2. Example 2

The following example, which has the same initial conditions, dynamic force, uniform storey stiffness, floor masses given in the Example 1, refers to three-storey structure shown in Fig. 5. The viscous damper has been moved on the top floor. All steps of analysis in previous example have been repeated and only responses of the second floor are discussed here.

According to Table 7, DOEA and MSM-b methods give the closest results to SSM method. It is proved again, that DOEA method is in the first order. While the maximum displacement response calculated by SSM and DOEA methods occurs in approximately 9 seconds, it occurs in 3.819 seconds with other approximate methods. The displacement response at 3.819 seconds of SSM method is seen to be 0.4819 m

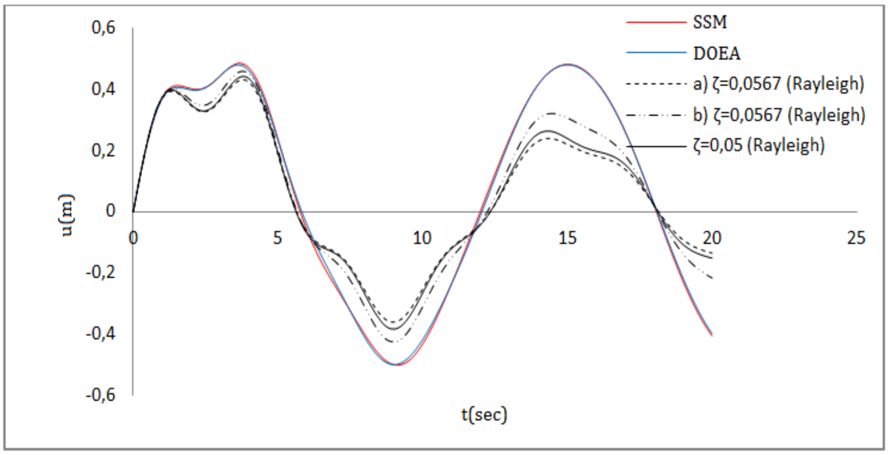

Fig. 6. Displacement response of second floor: (a) Rayleigh damping is created using first and second modes; (b) Rayleigh damping is created using first and third modes of vibration.

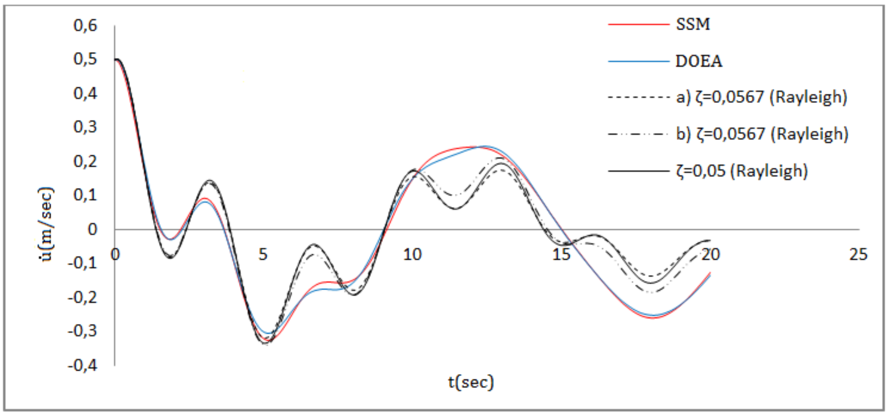

Fig. 7. Velocity response of second floor: (a) Rayleigh damping is created using first and second modes; (b) Rayleigh damping is created using first and third modes of vibration.

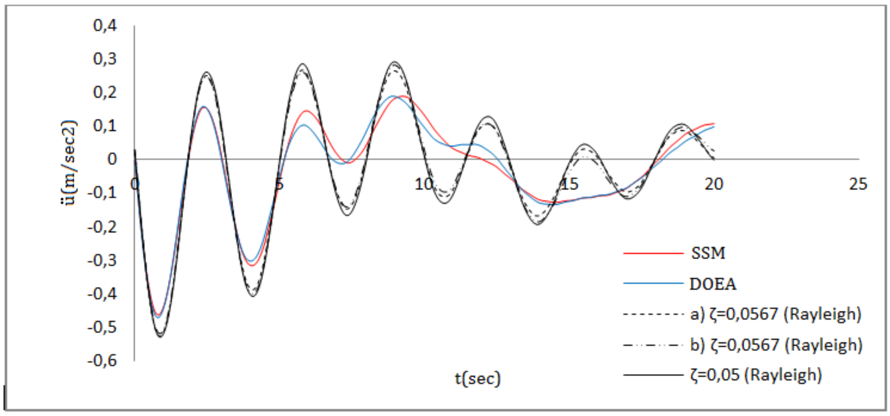

Fig. 8. Acceleration response of second floor: (a) Rayleigh damping is created using first and second modes; (b) Rayleigh damping is created using first and third modes of vibration.

Table 7. Maximum responses of the second floor.

Method	Time (sec)	Max. Dis. (m)	Time (sec)	Max. Vel. (m/sec)	Time (sec)	Max. Acce. (m/sec^2)
SSM	9.146 (3.819)	0.5011 (0.4819)	0	0.5	0.804	0.4623
DOEA	9.045	0.4993	0	0.5	0.804	0.4703
MSM-a	3.819	0.4303	0	0.5	0.9045	0.5246
MSM-b	3.819	0.4581	0	0.5	0.9045	0.5169
MSM-5%	3.819	0.4413	0	0.5	0.804	0.5244

As seen in Table 8, the exact modal damping ratio and natural frequency of the second mode are respectively 0.0427 and 1.4396. Among four approximate methods, the closest results to the exact solution, which are calculated by DOEA method, were found to be 0.0471 and 1.4142. As seen in the time history response of second floor, the closest response has been given by the DOEA method.

From Table 9, it is seen that the closest result to the exact solution is obtained by the DOEA and MSM-b method. Therefore, among these approximate methods the maximum deviation was found to be 16.24% by MSM-5%. Natural frequencies and modal damping ratios of the three-storey frame building with dampers at different floors were calculated and compared with each other. Being small natural frequencies, the frame building with a damper at the first floor has been tested to be more rigid than that with a damper at the top floor. Therefore, it would be better to put the damper at the first floor. Furthermore, when replacing the damper of the frame building from bottom to top floor, the damping matrix and natural frequencies of them are respectively changed by SSM and DOEA methods, but by other three methods no change has been seen on natural frequencies and damping matrix.

Table 8. Modal damping ratios and mode frequencies of the second mode.

Method	ζ_2	ω_2	ωd_2
SSM	0.0427	1.4396	1.4383
DOEA	0.0471	1.4142	1.4126
MSM-a	0.0567	1.4142	1.4119
MSM-b	0.0491	1.4142	1.4125
MSM-5%	0.0500	1.4142	1.4124

Table 9. Maximum responses of the frame building.

Method	Max. Dis. (m)	Max. Vel. (m/sec)	Max. Accel. (m/sec²)
SSM	0.7823	0.5093	0.7367
DOEA	0.7826	0.5091	0.7347
MSM-a	0.6871	0.5067	0.8424
MSM-b	0.7254	0.5068	0.8408
MSM-5%	0.6975	0.5071	0.8564

3.3. Example 3

The following example refers to a five-storey structure, which has non-uniform storey stiffness, floor masses m_1 and m_2, and two viscous dampers in the first and second storey of the structure, shown in Fig. 9. Results are discussed in comparative tables. The comparative vibrational graphs of each storey exist in Rasa (2017).

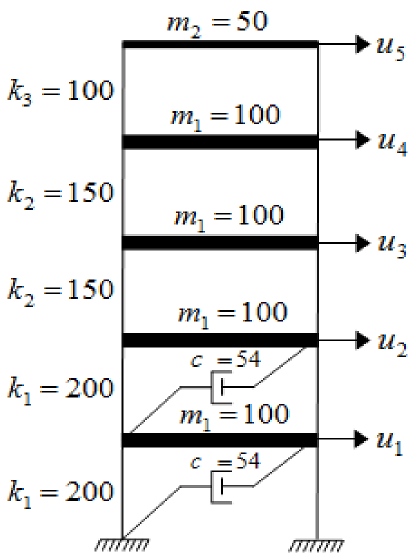

Fig. 9. Five-storey structure with two dampers at the first and second floors.

$$\boldsymbol{m} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 50 \end{bmatrix}$$

$$\boldsymbol{k} = \begin{bmatrix} 400 & -200 & 0 & 0 & 0 \\ -200 & 350 & -150 & 0 & 0 \\ 0 & -150 & 300 & -150 & 0 \\ 0 & 0 & -150 & 250 & -100 \\ 0 & 0 & 0 & -100 & 100 \end{bmatrix}$$

$$\boldsymbol{c} = \begin{bmatrix} 108 & -54 & 0 & 0 & 0 \\ -54 & 54 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{f}}(\boldsymbol{t}) = \begin{bmatrix} 10\sin\left(\frac{3}{2}t\right) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\boldsymbol{u}}(\boldsymbol{0}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix} \quad \underline{\dot{\boldsymbol{u}}}(\boldsymbol{0}) = \begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Mass matrix, damping matrix, and stiffness matrix of the structure, dynamic force and initial condition vector are given above. Comparative tables discussing the exact and approximate modal damping factors, damped frequencies and maximum responses of the system are given in detail. Creating proportional damping matrix of frame building needs a damping factor ζ, which is given approximately 5% and 0.081%. Taking the average of exact modal damping factors from Table 11 gives damping factor (0.081%).

Table 10. Exact and approximate mode frequencies of the building.

SSM	ν_n	DOEA and MSM	ω_n
ν_1	$-0.6391 \pm i2.2924$	ω_1	2.4915
ν_2	$-0.0437 \pm i2.1911$	ω_2	2.1246
ν_3	$-0.0528 \pm i1.6933$	ω_3	1.6756
ν_4	$-0.0594 \pm i1.1384$	ω_4	1.1374
ν_5	$-0.0149 \pm i0.4210$	ω_5	0.4205

As seen in Table 11, the closest result to the exact modal damping ratio of the building is given by DOEA method. The modal damping ratios obtained by other four approximate methods are smaller than that of exact method. Furthermore, a big difference between modal damping ratios has been seen in the fourth and fifth modes of vibration. In Table 12, the maximum responses calculated by DOEA method are very close to that of exact method as well. The most deviation among five approximate methods, which is calculated 38.51%, occurs in the MSM-a method.

Table 11. Exact and approximate modal damping ratios of the building.

Method	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
SSM	0.2686	0.0200	0.0312	0.0521	0.0353
DOEA	0.2133	0.0644	0.0403	0.0523	0.0354
MSM-a	0.0810	0.0810	0.0848	0.1016	0.2283
MSM-b	0.0810	0.0772	0.0752	0.0810	0.1598
MSM-c	0.0810	0.0795	0.0810	0.0935	0.2011
MSM-5%	0.0500	0.0500	0.0524	0.0627	0.1409

Table 12. Maximum responses of the frame building.

Method	Max. Dis. (m)	Max. Vel. (m/sec)	Max. Accel. (m/sec²)
SSM	0.3941	0.3409	0.6119
DOEA	0.2933	0.3597	0.4919
MSM-a	0.2423	0.2970	0.4304
MSM-b	0.2571	0.3131	0.4338
MSM-c	0.2480	0.3029	0.4312
MSM-5%	0.2754	0.3382	0.4764

3.4. Example 4

In this example the damper from second floor has been replaced on the fifth floor of the structure, and the same steps in Example 3 has been repeated here. The comparative vibrational graphs of each storey exist in Rasa (2017).

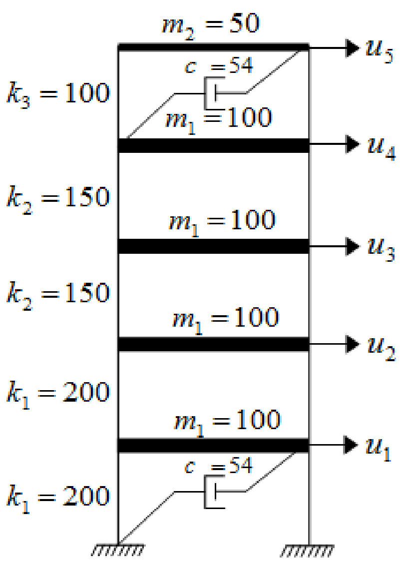

Fig. 10. Five-storey structure with two dampers at the first and fifth floors.

$$c = \begin{bmatrix} 54 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & -54 \\ 0 & 0 & 0 & -54 & 54 \end{bmatrix}$$

Table 13. Exact and approximate mode frequencies of the building.

SSM	v_n	DOEA and MSM	ω_n
v_1	$-0.1065 \pm i2.4574$	ω_1	2.4915
v_2	$-0.1653 \pm i1.9460$	ω_2	2.1246
v_3	$-0.6958 \pm i1.6449$	ω_3	1.6756
v_4	$-0.1033 \pm i1.1692$	ω_4	1.1374
v_5	$-0.0090 \pm i0.4212$	ω_5	0.4205

As seen in Table 14, the closest result to the exact modal damping ratios of the frame building is calculated by DOEA method. The modal damping ratios obtained by other five approximate methods (MSM-a, MSM-b, MSM-c, and MSM-5%) are seen to be larger than that of exact method. In approximate methods, fourth and fifth modal damping ratios are found to be larger than that of exact method. In Table 15, the maximum responses (0.2490 m, 0.294 m/sec and 0.4391m/sec2) which, are calculated by DOEA method, are close to that of exact method. Among these five approximate methods (DOEA, MSM-a, MSM-b, MSM-c, and MSM-5%), the most deviation occurs in MSM-a method with the rate of 15.13%. Due to the high rigidity and small natural frequencies of the frame building, the dampers could be placed in the first and fifth floors.

Table 14. Exact and approximate modal damping ratios of the building.

Method	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
SSM	0.0433	0.0846	0.3896	0.0880	0.0214
DOEA	0.0473	0.1889	0.2588	0.0216	0.1039
MSM-a	0.1253	0.1253	0.1312	0.1572	0.3531
MSM-b	0.1253	0.1194	0.1163	0.1253	0.2472
MSM-c	0.1253	0.1230	0.1253	0.1446	0.3112
MSM-5%	0.0500	0.0500	0.0524	0.0627	0.1409

Table 15. Maximum responses of the frame building.

Method	Max. Dis. (m)	Max. Vel. (m/sec)	Max. Accel. (m/sec²)
SSM	0.2553	0.3043	0.4785
DOEA	0.2490	0.2940	0.4391
MSM-a	0.2169	0.2665	0.4061
MSM-b	0.2304	0.2769	0.4080
MSM-c	0.2211	0.2694	0.4068
MSM-5%	0.2754	0.3382	0.4764

4. Conclusions

In this article, a comparative research has been conducted between the exact and approximate solution of structural vibration of multi storey buildings. The procedure has been implemented utilizing state-space method and mode superposition method, and illustrated by four examples. Considering the dynamical parameters and vibrations of frame buildings, the following results have been obtained:

- In approximate methods (DOEA, MSM-a, MSM-b, MSM-c, and MSM-5%), the maximum responses have generally occurred in the steady-state vibration part while in exact method (SSM) the maximum responses have occurred in the transient vibration part.
- Among five approximate methods (DOEA, MSM-a, MSM-b, MSM-c, MSM-5%), the closest response to the exact method was given by the Disregarding the Off-diagonal Elements Approach (DOEA).
- As it was noticed, the results obtained by mode superposition method were not very realistic when the damping ratio of the system was assumed an estimated value (such as 5%). The most deviation occurred in this case was 16.24%.
- In order to obtain better result with mode superposition method; It is recommended to use the average of damping factors, which are calculated by SSM method, as damping factor of frame building.
- It is recommended to use the first and last modes of vibrations when constructing the proportional damping matrix. The results obtained in this case are less accurate than DOEA method.

Damping mechanism of real structures is not a completely solved topic. Therefore, this research can be extended to detect the most suitable type of damping mechanism which can represent the real or near to the real behavior of structures.

In order to validate the results obtained by these methods, an experimental analysis of three and five storey frame buildings or a theoretical research utilizing other types of proportional/non-proportional damping matrix in the system, could be carried out as a future research problem. These observations could be emphasized applying earthquake loads on the system.

References

Avcar M (2104). Free vibration analysis of beams considering different geometric characteristics and boundary conditions. *International Journal of Mechanics and Applications*, 4(3), 94-100.

Borino G, Muscolin G (1986). Mode-Superposition methods in dynamic analysis of classically and non-classically damped linear systems. *Earthquake Engineering and Structural Dynamics*, 14, 705-717.

Caughey TK (1960). Classical normal modes in damped linear dynamic systems. *Journal of Applied Mechanics*, 27, 269-271.

Caughey TK, O'Kelly M (1965). Classical normal modes in damped linear dynamic system. *Journal of Applied Mechanics*, 32(3), 583-588.

Civalek O, Avcar M (2020). Free vibration and buckling analyses of CNT reinforced laminated non-rectangular plates by discrete singular convolution method. *Engineering with Computers.*

Felszeghy SF (1993). On uncoupling and solving the equations of motion of vibrating linear discrete systems. *Transactions of the ASME*, 60(2), 456-462.

Rasa AY (2017). Çok Katlı Yapıların Geçici Titreşimlerinin Durum-Uzayı Yaklaşımı ile İncelenmesi ve Modların Birleştirilmesi Yöntemiyle bir Karşılaştırma. *M.Sc. thesis*, Atatürk University, Erzurum, Turkey. (in Turkish)

Sinha R, Igusa T (1992). CQC and SRSS methods for non-classically damped structures. *Earthquake Engineering and Structural Design,* 24, 615-619.

Veletsos AS, Ventura CE (1986). Modal analysis of nonclassically damped linear systems. *Earthquake Engineering and Structure Dynamics,* 14, 217-243.

Villaverde R (2008). A complex modal superposition method for the seismic analysis of structures with supplemental dampers. *The 14th World Conference on Earthquake Engineering*, 12-17, Beijing, China.

Wilson EL, Kiureghian A, Bayo EP (1981). A replacement for the SRSS method in seismic analysis. *Earthquake Engineering and Structural Dynamics*, 9, 187-192.

Wilson EL, Penzien J (1981). Evaluation of orthogonal damping matrices. *International Journal for Numerical Methods in Engineering,* 4, 5-10.

Zhou XY, Yu RF, Liang D (2004). The complex-complete-quadratic combination (CCQC) method for seismic responses of non-classically damped linear MDOF system. *13th World Conference on Earthquake Engineering Vancouver*, B.C., Canada, paper No. 848.